

Calculations for the Greenhouse Development Rights Calculator

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Abstract

The Greenhouse Development Rights (GDRs) on-line calculator is a web-based tool for examining the implications of different assumptions under the GDRs framework. The calculator reports a number of indicators that require technical documentation.

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Introduction

The Greenhouse Development Rights (GDRs) Framework (Baer et al., 2007) is a general framework for burden-sharing for climate change that takes both responsibility and financial capacity into account. The framework takes the individual as the unit of analysis, which means that at a minimum instantaneous income distributions within countries are a necessary input into calculations that make use of the framework. The availability, since the mid-1990s, of a reasonably complete international database on income distribution (Deininger and Squire, 1996; UNU-WIDER, 2008) makes this calculation possible. While income mobility – the trajectory of individuals over time within an income distribution – would also be useful, there are insufficient data on income mobility to support an international calculation.

An on-line calculator is being developed that is consistent with the GDRs framework and that is used by the developers of the GDRs framework in order to explore the implications of the framework at national and individual level.[1] This working paper describes some of the calculations that are implemented in the calculator.

Lognormal Income Distributions

Following Kemp-Benedict (2001) and Lopez and Servén (2006), income is assumed to be distributed lognormally within countries. The lognormal distribution has two parameters: the mean income $\bar{y}$ and the standard deviation of the log of income, σ. The standard deviation of the log of income is a measure of how equally or unequally distributed income is within the country, and can be related to the well-known Gini coefficient *G* using the following formula,

$$\sigma = \sqrt{2}\, N^{-1}\left(\frac{1+G}{2}\right). \tag{1}$$

In this expression, N^{-1} is the inverse normal function.

It is convenient when working with the distribution within an individual country to transform income *y* into a new variable *z*, given by

$$z = \frac{1}{\sigma}\ln(y/\bar{y}) + \frac{\sigma}{2}. \tag{2}$$

In terms of *z*, the lognormal becomes simply a normal distribution,

$$y \sim \text{Lognormal}(\bar{y}, \sigma) \;\Rightarrow\; z \sim N(0,1). \tag{3}$$

Partial Moments of the Lognormal Income Distribution

The GDRs framework proposes that income below a certain lower income threshold y_l is exempt, in that it does not contribute to an individual's capacity to pay. Furthermore, in the practical implementation of the framework within the calculator, it is assumed that emissions within countries vary with income with an elasticity that is the same for all countries, and emissions associated with incomes below y_l are exempt, in that they do not contribute to an individual's responsibility.

1 The calculator is under development. It is available on a password-protected web site, and at present access is very limited. Please contact the author of this working paper to inquire about accessing the calculator.

More specifically, individual per capita emissions are assumed to vary in the following way with income,

$$\epsilon_i(y) = A_i y^\gamma \,, \tag{4}$$

where $\epsilon_i(y)$ is emissions per capita for individuals in a narrow range of incomes around y for country i, and A_i is a country-specific constant. The elasticity γ is the same for all countries. Chakravarty et al. (in preparation) report evidence that this assumption stands up reasonably well to empirical test, and that the data are consistent with γ between about 0.7 and 1.0.

In order to implement the framework, it is necessary to evaluate integrals of the following form, which are referred to in this working paper as "partial moments",

$$M_\gamma(y; y_l, \sigma, \bar{y}) \equiv \int_{y_l}^{\infty} dy \, \left(y^\gamma - y_l^\gamma\right) f(y; \bar{y}, \sigma) \,, \tag{5}$$

where $f(y; \bar{y}, \sigma)$ is the (lognormal) income distribution. Using the convenient change of variables given in Equation (2), for a lognormal distribution the partial moments can be shown to be equal to

$$M_\gamma(y; y_l, \sigma, \bar{y}) = \bar{y}^\gamma e^{\frac{\sigma^2}{2}\gamma(\gamma-1)} \left(1 - N(z_l - \gamma\sigma)\right) - y_l^\gamma \left(1 - N(z_l)\right) , \tag{6}$$

where N is the cumulative normal distribution. This follows from the following equation,

$$\int_{y_l}^{\infty} dy \; y^\gamma f(y; \sigma, \bar{y}) \;=\; \bar{y}^\gamma e^{-\gamma\sigma^2/2} \int_{z_l}^{\infty} dz \; e^{\gamma\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \;=\; \bar{y}^\gamma e^{\frac{\sigma^2}{2}\gamma(\gamma-1)} \left(1 - N(z_l - \gamma\sigma)\right) . \tag{7}$$

Capacity and Responsibility

In principle it is possible to have more than one income threshold and have different fractions of income contribute toward capacity to different degrees at different income levels. In some versions of the calculator this is implemented using two different thresholds, a lower threshold y_l, and an upper threshold y_u. Income is exempt below the lower threshold, while above the upper threshold, 100% of income contributes to capacity. Between the two thresholds, a fraction φ of each marginal increment of income contributes toward capacity. The definitions of capacity and responsibility will be given for this general case.

Individual capacity and responsibility per capita

Individual capacity per capita in country i, $c_i(y)$, is given by

$$c_i(y) = \begin{cases} 0 & y < y_l \\ \varphi(y - y_l) & y_l \leqslant y \leqslant y_u \\ \varphi(y_u - y_l) + (y - y_u) & y > y_u \end{cases} . \tag{8}$$

Responsibility is based on emissions above the threshold, where individual emissions per capita vary as in Equation (4). The annual contribution to individual per capita responsibility r_i^{ann} is then given by

$$r_i^{\text{ann}}(y)=\begin{cases} 0 & y<y_l \\ \varphi\, A_i(y^\gamma-y_l^\gamma) & y_l\leqslant y\leqslant y_u \\ A_i\left[\varphi(y_u^\gamma-y_l^\gamma)+(y^\gamma-y_u^\gamma)\right] & y>y_u \end{cases}. \quad (9)$$

The difficulty with Equation (9) is that the coefficient A_i is not known. What is known is the total national annual emissions E_i, given by

$$E_i = \int_0^\infty dy\ A_i y^\gamma f(y;\bar{y}_i,\sigma_i)=A_i\,\bar{y}_i^{\,\gamma}\,e^{\frac{\sigma_i^2}{2}\gamma(\gamma-1)}. \quad (10)$$

As a result,

$$A_i=\frac{E_i}{\bar{y}_i^{\,\gamma}}\,e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}, \quad (11)$$

and

$$r_i^{\text{ann}}(y)=\begin{cases} 0 & y<y_l \\ \dfrac{E_i}{\bar{y}_i^{\,\gamma}}e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}\varphi(y^\gamma-y_l^\gamma) & y_l\leqslant y\leqslant y_u \\ \dfrac{E_i}{\bar{y}_i^{\,\gamma}}e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}\left[\varphi(y_u^\gamma-y_l^\gamma)+(y^\gamma-y_u^\gamma)\right] & y>y_u \end{cases}. \quad (12)$$

National capacity and responsibility

National capacity C_i is given by integrating per capita capacity [Equation (8)] over the income distribution. Using the formula for partial moments given in Equation (6), and rearranging, this can be shown to be equal to

$$C_i=\bar{y}\left[1-(1-\varphi)N(y_u-\sigma_i)-\varphi N(y_l-\sigma_i)\right]-(1-\varphi)y_u\left(1-N(y_u)\right)-y_l\left(1-N(y_l)\right). \quad (13)$$

Note that in the case φ = 1, this collapses to the expression for a single threshold,

$$\varphi=1\ \Rightarrow\ C_i=\bar{y}\left[1-N(y_l-\sigma_i)\right]-y_l\left(1-N(y_l)\right). \quad (14)$$

National annual responsibility R_i^{ann} is found by integrating per capita annual responsibility [Equation (12)] over the income distribution. This can be shown to equal

$$\begin{aligned} R_i^{\text{ann}} = {} & E_i\left[1-(1-\varphi)N(y_u-\gamma\sigma_i)-\varphi N(y_l-\gamma\sigma_i)\right] \\ & -E_i e^{-\frac{\sigma^2}{2}\gamma(\gamma-1)}\left\{(1-\varphi)\left(\frac{y_u}{\bar{y}}\right)^\gamma\left(1-N(y_u)\right)+\left(\frac{y_l}{\bar{y}}\right)^\gamma\left(1-N(y_l)\right)\right\}. \end{aligned} \quad (15)$$

As with capacity, this collapses to the one-threshold case when φ = 1.

In any given year *Y*, national responsibility R_i is calculated as cumulative annual responsibility from a starting year (currently, 1990). Adding a time index *t* to annual responsibility, this can be written

$$R_{i,Y} = \sum_{t=1990}^{Y} R_{i,t}^{\text{ann}} \,. \tag{16}$$

In subsequent formulas, the time index will be suppressed.

The Responsibility-Capacity Indicator

The Responsibility-Capacity Indicator (RCI) is the key indicator for the GDRs framework. It is used to allocate burden-sharing. National RCI, as currently implemented, is a weighted sum of the national share of global responsibility and capacity. That is,

$$\text{RCI}_i = a\frac{R_i}{\sum_{i=1}^{N} R_i} + (1-a)\frac{C_i}{\sum_{i=1}^{N} C_i}\,, \tag{17}$$

where N is the number of countries. National RCI sums to one across countries, and so national RCI is the national share of total RCI. For an individual within country i, RCI per capita, $\text{rci}_i(y)$, is calculated as

$$\text{rci}_i(y) = a\frac{R_i}{\sum_{i=1}^{N} R_i}\left(\frac{r_i^{\text{ann}}(y)}{R_i^{\text{ann}}}\right) + (1-a)\frac{c_i(y)}{\sum_{i=1}^{N} C_i}\,. \tag{18}$$

This formulation assumes that cumulative responsibility is allocated according to the current annual allocation in any given year. This is not ideal, since in fact individuals will have their own income trajectories that can be quite divergent, depending on the degree of income mobility within a country. However, given the very limited data on income mobility it is not feasible to try to capture this.

Burden Sharing as a Tax

RCI is used to allocate burden sharing, which can be thought of as a tax. The total tax T is given as a fraction τ of gross world output, GWP,

$$T = \tau\,\text{GWP}\,. \tag{19}$$

National tax T_i is then given by national RCI multiplied by T,

$$T_i = \text{RCI}_i\,T\,, \tag{20}$$

which sums to the total tax because national RCI sums to one across countries.

Individual per capita tax $t_i(y)$ is given by

$$t_i(y) = T\,\text{rci}_i(y)\,. \tag{20}$$

The Suits Index

The Suits index (Suits, 1977) is a commonly-used indicator of the degree of progressivity or regressivity of a tax or tax system. It is calculated by finding the cumulative share of the total tax across individuals and plotting against cumulative income, in a way that is analogous to the calculation of the Gini coefficient from the Lorenz curve. In the GDRs calculator the Suits index is an optional indicator that can be calculated for the world as a whole.

In order to calculate the Suits index, the global income distribution $f(y)$ is needed. This is given by

$$f(y)=\sum_{i=1}^{N} p_i f_i(y;\bar{y}_i,\sigma_i)\,, \tag{21}$$

where p_i is country i's share of global population. Note that $f(y)$ is not lognormal, although each of the country distributions is lognormal.

Using the global income distribution, the cumulative share of global income up to income y^*, $x(y^*)$, is given by

$$x(y^*) \;=\; \frac{1}{\langle y\rangle}\int_0^{y^*} dy\; y f(y) \;=\; \frac{1}{\langle y\rangle}\sum_{i=1}^{N} p_i \int_0^{y^*} dy\; y f_i(y;\bar{y}_i,\sigma_i)\,, \tag{22}$$

where $\langle y\rangle$ is global mean income, equal to $\lim_{y^*\to\infty} x(y^*)$.

The cumulative share of total tax $T(y^*)$ up to income y^* is given by

$$T(y^*)=\frac{1}{\langle T\rangle}\sum_{i=1}^{N} p_i \int_0^{y^*} dy\; t_i(y) f_i(y;\bar{y}_i,\sigma_i)\,, \tag{23}$$

where $t_i(y)$ is individual tax per capita, given by Equation (20). The set of points $(T(y^*), x(y^*))$ traces out a curve $T(x)$ as y^* goes from 0 to infinity that is analogous to the Lorenz curve. In terms of the curve $T(x)$, the Suits index is defined as (Suits, 1977)

$$S=1-2\int_0^1 dx\; T(x)\,. \tag{24}$$

Because x is known as a function of income, $x(y^*)$, this can be re-written

$$S=1-2\int_0^{\infty} dy^*\; \frac{dx}{dy^*} T(y^*) \tag{25}$$

by a change of variables. It is more convenient to work with the derivative of $T(y^*)$ than with the derivative of $x(y^*)$. This can be accomplished by integrating Equation (25) by parts, to give

$$S=1-2\,x(y^*)T(y^*)\Big|_{y^*=0}^{y^*=\infty}+2\int_0^{\infty} dy^*\; x(y^*)\frac{dT(y^*)}{dy^*}\,, \tag{26}$$

but since $x(y^*)$ and $T(y^*)$ go from 0 to 1, this becomes simply

$$S=2\int_0^{\infty} dy^*\; x(y^*)\frac{dT(y^*)}{dy^*}-1\,. \tag{27}$$

Using Equation (22) and the formula for partial moments [Equation (6)], $x(y^*)$ can be shown to be equal to

$$x(y^*)=\frac{1}{\langle y\rangle}\sum_{i=1}^{N} p_i \bar{y}_i N(z_i^*-\sigma_i)\,, \tag{28}$$

where z_i^* is the value for z corresponding to y^* using Equation (2). The other factor in the integral in Equation (27) is $dT(y^*)/dy^*$, which can be calculated directly from Equation (23) as

$$\frac{dT(y^*)}{dy^*} = \frac{1}{\langle T \rangle} \sum_{i=1}^{N} p_i t_i(y^*) f_i(y^*; \bar{y}_i, \sigma_i) \,. \tag{29}$$

With Equation (28) and (29), the Suits index can be calculated by numerically evaluating the integral in Equation (27).

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